THE CHARGED PENROSE INEQUALITY FOR A SPHERICALLY SYMMETRIC BLACK HOLE (Geometry of Moduli Space of Low Dimensional Manifolds)

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THE CHARGED PENROSE INEQUALITY FOR A SPHERICALLY SYMMETRIC BLACK HOLE

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ABSTRACT. We give an alternate proof of the charged Penrose inequality for a spherically symmetric black hole, in the non-time-symmetric case.

1. INTRODUCTION

Consider an initial data set (M, g, k, E) for the Einstein-Maxwell equations with vanishing magnetic field. Here M is a Riemannian 3-manifold with metric g, k is a symmetric 2-tensor representing the second fundamental form of the embedding into spacetime, and E denotes the electric field. It is assumed that the manifold has a boundary ∂M consisting of an outermost apparent horizon. That is, if H denotes mean curvature with respect to the normal pointing towards spatial infinity, then each boundary component $S \subset \partial M$ satisfies $\theta_+(S) := H_S + Tr_S k = 0$ (future horizon) and/or $\theta_-(S) := H_S - Tr_S k = 0$ (past horizon), and there are no other apparent horizons present. Moreover the data are taken to be asymptotically flat with one end, in that outside a compact set the manifold is diffeomorphic to the complement of a ball in $\mathbb{R}^3$, and in the coordinates given by this asymptotic diffeomorphism the following fall-off conditions hold

$$|\partial^m(g_{ij} - \delta_{ij})| = O(|x|^{-m-1}), \quad |\partial^m k_{ij}| = O(|x|^{-m-2}), \quad |\partial^m E^i| = O(|x|^{-m-2}), \quad m = 0, 1, 2, \quad \text{as } |x| \to \infty.$$

With a vanishing magnetic field, the matter and current densities for the non-electromagnetic matter fields are given by

$$2\mu = R + (Trk)^2 - |k|_g^2 - 2|E|_g^2,$$
$$J = div(k - (Trk)g),$$

where R denotes the scalar curvature of g. The following inequality will be referred to as the dominant energy condition

$$\mu \geq |J|_g. \tag{1.1}$$

Under these hypotheses and based on heuristic arguments of Penrose [9] which rely heavily on the cosmic censorship conjecture, the following inequality relating the ADM energy and the minimal area $\mathcal{A}$ required to enclose the boundary ∂M, has been conjectured to hold [4, 8]

$$E_{ADM} \geq \sqrt{\frac{\mathcal{A}}{16\pi}} + \sqrt{\frac{\pi}{\mathcal{A}}} Q^2, \tag{1.2}$$

where $Q = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S_r} E^i \nu_i$ is the total electric charge, with S_r coordinate spheres in the asymptotic end having unit outer normal ν. Inequality (1.2) has been proven by Jang [8] for time-symmetric initial data with a connected horizon, under the assumption that a smooth solution to the Inverse Mean Curvature Flow (IMCF) exists. Moreover in light of Huisken and Ilmanen's work [7], the hypothesis of a smooth IMCF can be discarded. However without the assumption of a connected horizon, counterexamples [11] are known to exist; these examples do not provide a contradiction to the cosmic censorship conjecture. In the non-time-symmetric case this inequality has been proven under the additional hypothesis of spherically symmetric

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initial data [6]. In the general case, with a connected horizon, the validity of (1.2) has been reduced to solving a coupled system of equations involving the generalized Jang equation and the IMCF [3]. In the case of equality, it is expected that the initial data arise from the Reissner-Nordström spacetime; this has been confirmed in the time-symmetric case [3].

In this note we give an alternate proof (in the non-time-symmetric case) of inequality (1.2) as well as the rigidity statement, under the assumption of spherical symmetry. The proof relies on the generalized Jang equation.

Theorem 1.1. *Let (M, g, k, E) be a 3-dimensional, spherically symmetric, asymptotically flat initial data set for the Einstein-Maxwell system with an outermost apparent horizon boundary ∂M. Assume that the charge density is zero $divE = 0$, that the magnetic field vanishes, and that the non-electromagnetic matter fields satisfy the dominant energy condition* (1.1). *Then*

$$E_{ADM} \geq \sqrt{\frac{|\partial M|}{16\pi}} + \sqrt{\frac{\pi}{|\partial M|}}Q^2, \tag{1.3}$$

and equality implies that the initial data arise from the Reissner-Nordström spacetime.

In the case of spherical symmetry, inequality (1.2) is equivalent to inequality (1.3). To see this, observe that in spherical symmetry the outermost apparent horizon assumption implies that M is foliated by surfaces of positive mean curvature. Therefore ∂M is outerminimzing, and $|\partial M| = \mathcal{A}$.

2. Charged Jang Deformation

In the time-symmetric case when $k = 0$, the dominant energy condition (1.1) reduces to

$$R \geq 2|E|_g. \tag{2.1}$$

This inequality is heavily relied upon in the proof of the charged Penrose inequality [8]. In fact the main difficulty in extending the proof to the non-time-symmetric case, is the lack of this inequality under the dominant energy condition assumption. For this reason we seek a deformation of the initial data to a new set $(\Sigma, \overline{g}, \overline{E})$, where Σ is diffeomorphic to M, and the metric $\overline{g}$ and vector field $\overline{E}$ are related to g and E in a precise way described below. The purpose of the deformation is to obtain new initial data which satisfy (2.1) in a weak sense, while preserving all other quantities appearing in the charged Penrose inequality, such as the charge density, total charge, ADM energy, and boundary area.

Consider the warped product 4-manifold $(M \times \mathbb{R},\, g + \phi^2 dt^2)$, where ϕ is a nonnegative function to be chosen appropriately. Let $\Sigma = \{t = f(x)\}$ be the graph of a function f inside this warped product setting, then the induced metric on Σ is given by $\overline{g} = g + \phi^2 df^2$. In [1, 2] it is shown that in order to obtain the most desirable positivity property for the scalar curvature of the graph, the function f should satisfy

$$\left(g^{ij} - \frac{\phi^2 f^i f^j}{1 + \phi^2|\nabla f|_g^2}\right)\left(\frac{\phi\nabla_{ij}f + \phi_i f_j + \phi_j f_i}{\sqrt{1 + \phi^2|\nabla f|_g^2}} - k_{ij}\right) = 0, \tag{2.2}$$

where ∇ denotes covariant differentiation with respect to the metric g, $f_i = \partial_i f$, and $f^i = g^{ij}f_j$. Equation (2.2) is referred to as the generalized Jang equation, and when it is satisfied Σ will be called the Jang surface. This equation is quasi-linear elliptic, and degenerates when either $\phi = 0$ or f blows-up. The existence, regularity, and blow-up behavior for the generalized Jang equation is studied at length in [5]. The scalar curvature of the Jang surface [1, 2] is given by

$$\overline{R} = 2(\mu - J(w)) + 2|E|_{\overline{g}}^2 + |h - k|_{\overline{g}}^2 + 2|q|_{\overline{g}}^2 - 2\phi^{-1}\overline{div}(\phi q), \tag{2.3}$$

where $\overline{div}$ is the divergence operator with respect to $\overline{g}$, h is the second fundamental form of the graph $t = f(x)$ in the Lorentzian 4-manifold $(M \times \mathbb{R},\, \overline{g} - \phi^2 dt^2)$, and w and q are 1-forms given by

$$h_{ij} = \frac{\phi \nabla_{ij} f + \phi_i f_j + \phi_j f_i}{\sqrt{1 + \phi^2 |\nabla f|_g^2}}, \qquad w_i = \frac{\phi f_i}{\sqrt{1 + \phi^2 |\nabla f|^2}}, \qquad q_i = \frac{\phi f^j}{\sqrt{1 + \phi^2 |\nabla f|^2}}(h_{ij} - k_{ij}).$$

If the dominant energy condition is satisfied, then all terms appearing on the right-hand side of (2.3) are nonnegative, except possibly the last term. However the last term is a divergence, which in many cases can be 'integrated away' when ϕ is chosen appropriately, so that in effect the scalar curvature is weakly nonnegative (that is, nonnegative when integrated against certain functions). For the charged Penrose inequality, a stronger condition than simple nonnegativity is required, more precisely we seek an inequality (holding in the weak sense) of the following form

$$\overline{R} \geq 2|\overline{E}|_{\overline{g}}^2, \tag{2.4}$$

where $\overline{E}$ is an auxiliary electric field defined on the Jang surface. This auxiliary electric field is required to satisfy three properties, namely

$$|E|_g \geq |\overline{E}|_{\overline{g}}, \qquad \overline{div}\, \overline{E} = 0, \qquad \overline{Q} = Q, \tag{2.5}$$

where $\overline{Q}$ is the total charge defined with respect to $\overline{E}$. In particular, if the first inequality of (2.5) is satisfied, then the dominant energy condition (1.1) and the scalar curvature formula (2.3) imply that (2.4) holds weakly. It turns out that there is a very natural choice for this auxiliary electric field, namely $\overline{E}$ is the induced electric field on the Jang surface Σ arising from the field strength F of the electromagnetic field on $(M \times \mathbb{R},\, g + \phi^2 dt^2)$. More precisely $\overline{E}_i = F(N, X_i)$, where N and X_i are respectively the unit normal and canonical tangent vectors to Σ

$$N = \frac{\phi^{-1}\partial_t - \phi f^i \partial_i}{\sqrt{1 + \phi^2 |\nabla f|_g^2}}, \qquad X_i = \partial_i + f_i \partial_t,$$

and $F = \frac{1}{2} F_{ab} dx^a \wedge dx^b$ is given by $F_{0i} = \phi E_i$ and $F_{ij} = 0$ for $i = 1,2,3$, with x^i, $i = 1,2,3$ coordinates on M and $x^0 = t$. In matrix form

$$F = \begin{pmatrix} 0 & \phi E_1 & \phi E_2 & \phi E_3 \\ -\phi E_1 & 0 & 0 & 0 \\ -\phi E_2 & 0 & 0 & 0 \\ -\phi E_3 & 0 & 0 & 0 \end{pmatrix}.$$

In [3] it is shown that

$$\overline{E}_i = \frac{E_i + \phi^2 f_i f^j E_j}{\sqrt{1 + \phi^2 |\nabla f|_g^2}},$$

and that all the desired properties of (2.5) hold.

When f solves (2.2) and $\overline{E}$ is given by (2), the triple $(\Sigma, \overline{g}, \overline{E})$ is referred to as charged Jang initial data.

In order to apply these constructions to the charged Penrose inequality, we need not only the (weak version of) inequality (2.4), but also three other properties of the charged Jang initial data. Let $S_0 \subset M$ denote the outermost minimal area enclosure of ∂M, and let $\mathcal{S}_0$ be the vertical lift of S_0 to Σ. Then the desired three properties are

$$\overline{E}_{ADM} = E_{ADM}, \qquad |\mathcal{S}_0|_{\overline{g}} = |S_0|_g =: \mathcal{A}, \qquad \overline{H}_{\mathcal{S}_0} = 0, \tag{2.6}$$

where $\overline{E}_{ADM}$ is the ADM energy of the Jang metric $\overline{g}$, and $|\mathcal{S}_0|_{\overline{g}}$ and $\overline{H}_{\mathcal{S}_0}$ are the area and mean curvature of the surface $\mathcal{S}_0$, respectively. The first of these equalities is achieved by imposing zero Dirichlet boundary conditions for f at spatial infinity. More precisely, if

$$\phi(x) = 1 + \frac{C}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as} \quad |x| \to \infty$$

for some constant C, then according to [5]

$$|\nabla^m f|(x) = O(|x|^{-\frac{1}{2}-m}) \quad \text{as} \quad |x| \to \infty, \qquad m = 0, 1, 2,$$

which is enough to ensure that the two ADM energies agree. The second equality of (2.6) may be obtained by imposing zero Dirichlet boundary conditions for the warping factor $\phi|_{S_0} = 0$. Notice that this conclusion should hold whether f blows-up or does not blow-up at S_0, since when blow-up occurs the Jang surface asymptotically approaches a cylinder over the blow-up region. It is well-known that the Jang surface can only blow-up on the portion of S_0 which coincides with the apparent horizon boundary. Lastly, the third equality of (2.6) is considered to be an appropriate boundary condition for the solutions of the generalized Jang equation (2.2). Typically, on the portion of S_0 which coincides with the apparent horizon boundary, this boundary condition forces the solution f to blow-up as just described, however this is not always the case.

3. Proof of Theorem 1.1

Let $(\Sigma, \overline{g}, \overline{E})$ be charged Jang initial data for some choice of warping factor ϕ, with $\phi|_{S_0} = 0$. Since the original initial data are spherically symmetric, the outermost apparent horizon assumption implies that the outermost minimal area enclosure S_0 agrees with the boundary ∂M. If the function ϕ vanishes appropriately at ∂M, then the Jang surface Σ is a manifold with boundary, moreover its boundary is the vertical lifting $\mathcal{S}_0$ which is outerminimizing, since $\overline{g} \geq g$. As Σ is also spherically symmetric, there then exists a smooth IMCF $\{\mathcal{S}_\tau\}_{\tau=0}^{\infty} \subset \Sigma$ starting from $\partial\Sigma$.

Consider the charged Hawking mass ([3], [6])

$$M_{CH}(\mathcal{S}_\tau) = \sqrt{\frac{|\mathcal{S}_\tau|_{\overline{g}}}{16\pi}} \left(1 + \frac{4\pi Q^2}{|\mathcal{S}_\tau|_{\overline{g}}} - \frac{1}{16\pi}\int_{\mathcal{S}_\tau} \overline{H}^2\right).$$

Standard properties of the IMCF [7] imply that

$$\frac{d}{d\tau} M_{CH}(\mathcal{S}_\tau) = -\frac{1}{2}\sqrt{\frac{\pi}{|\mathcal{S}_\tau|_{\overline{g}}}} Q^2 + \frac{1}{16\pi}\sqrt{\frac{|\mathcal{S}_\tau|_{\overline{g}}}{16\pi}} \int_{\mathcal{S}_\tau} \left(2\frac{|\nabla_{\mathcal{S}_\tau}\overline{H}|^2}{\overline{H}^2} + |\overline{A}|^2 - \frac{1}{2}\overline{H}^2 + \overline{R}\right), \tag{3.1}$$

where $\overline{A}$ and $\overline{H}$ are, respectively, the second fundamental form and mean curvature of $\mathcal{S}_\tau$. Since

$$|\overline{A}|^2 - \frac{1}{2}\overline{H}^2 = \frac{1}{2}(\lambda_1 - \lambda_2)^2,$$

where λ_i, $i = 1, 2$, are the principal curvatures of $\mathcal{S}_\tau$, this term is nonnegative. Therefore (3.1) combined with (2.3) gives

$$\frac{d}{d\tau} M_{CH}(\mathcal{S}_\tau) \geq -\frac{1}{2}\sqrt{\frac{\pi}{|\mathcal{S}_\tau|_{\overline{g}}}} Q^2 + \frac{1}{16\pi}\sqrt{\frac{|\mathcal{S}_\tau|_{\overline{g}}}{16\pi}} \int_{\mathcal{S}_\tau} \left(2|E|_g^2 - \frac{2}{\phi}\overline{div}(\phi q)\right)$$

where the dominant energy condition (1.1) and the fact that $|w|_g \leq 1$ have been used. From (2.5) and Hölder's inequality it follows that

$$\int_{\mathcal{S}_\tau} |E|_g^2 \geq \int_{\mathcal{S}_\tau} |\overline{E}|_{\overline{g}}^2 \geq \int_{\mathcal{S}_\tau} \langle \overline{E}, \nu_{\overline{g}} \rangle^2 \geq \frac{\left(\int_{\mathcal{S}_\tau} \langle \overline{E}, \nu_{\overline{g}} \rangle\right)^2}{|\mathcal{S}_\tau|_{\overline{g}}},$$

where $\nu_{\overline{g}}$ in the unit outer normal to $\mathcal{S}_\tau$. Applying the divergence theorem on the region $\Omega \subset \Sigma$ between $\mathcal{S}_\tau$ and spatial infinity, and using (2.5), produces

$$\int_{\mathcal{S}_\tau} \langle \overline{E}, \nu_{\overline{g}} \rangle = -\int_{\Omega} \overline{div}\,\overline{E} + 4\pi\overline{Q} = 4\pi Q.$$

Hence

$$\frac{d}{d\tau} M_{CH}(\mathcal{S}_\tau) \geq -\frac{1}{16\pi}\sqrt{\frac{|\mathcal{S}_\tau|_{\overline{g}}}{16\pi}} \int_{\mathcal{S}_\tau} \frac{2}{\phi}\overline{div}(\phi q). \tag{3.2}$$

The next step will be to integrate the above inequality between zero and infinity. Observe that since

$$M_{CH}(\mathcal{S}_\tau) = \sqrt{\frac{\pi}{|\mathcal{S}_\tau|_{\overline{g}}}} Q^2 + M_H(\mathcal{S}_\tau)$$

where M_H denotes the unaltered Hawking mass, and $|\mathcal{S}_\tau|_{\overline{g}}$ grows exponentially in τ, we have that

$$\lim_{\tau \to \infty} M_{CH}(\mathcal{S}_\tau) = \overline{E}_{ADM} = E_{ADM}.$$

On the other hand, since (by (2.6)) $\mathcal{S}_0$ is a minimal surface and $|\mathcal{S}_0|_{\overline{g}} = |\mathcal{S}_0|_g = |\partial M|$,

$$M_{CH}(\mathcal{S}_0) = \sqrt{\frac{|\partial M|}{16\pi}}\left(1 + \frac{4\pi Q^2}{|\partial M|}\right).$$

Therefore integrating (3.2) yields

$$E_{ADM} - \sqrt{\frac{|\partial M|}{16\pi}}\left(1 + \frac{4\pi Q^2}{|\partial M|}\right) \geq -\frac{2}{(16\pi)^{\frac{3}{2}}} \int_{\Sigma} \frac{\sqrt{|\mathcal{S}_\tau|_{\overline{g}}}\,\overline{H}}{\phi}\overline{div}(\phi q),$$

after applying the co-area formula. This suggests that we choose

$$\phi = \sqrt{\frac{|\mathcal{S}_\tau|_{\overline{g}}}{16\pi}}\,\overline{H}. \tag{3.3}$$

Note that it was shown in [1] that (assuming spherical symmetry) there is a smooth solution of the generalized Jang equation coupled to IMCF with this choice of ϕ, such that the desired properties (2.6) hold. We may then proceed to find

$$\frac{1}{\sqrt{16\pi}} \int_{\Sigma} \frac{\sqrt{|\mathcal{S}_\tau|_{\overline{g}}}\,\overline{H}}{\phi}\overline{div}(\phi q) = \int_{\Sigma} \overline{div}(\phi q) = \int_{\mathcal{S}_\infty} \phi\langle q, \nu_{\overline{g}}\rangle - \int_{\mathcal{S}_0} \phi\langle q, \nu_{\overline{g}}\rangle.$$

Well-known behavior of solutions to the (generalized) Jang equation [5, 10] shows that $q(x) \to 0$ as $|x| \to \infty$, and that q remains bounded on $\mathcal{S}_0$ even if the Jang surface blows-up over this surface. Moreover $\phi \to 1$ as $|x| \to \infty$ and $\phi = 0$ on $\mathcal{S}_0$, since $\overline{H}_{\mathcal{S}_0} = 0$ by (2.6). Hence both boundary integrals vanish, and this yields the inequality (1.3).

Suppose that equality holds in (1.3), then all inequalities appearing in this section must be equalities and the following quantities must vanish

$$\mu - J(w) = |h - k|_{\overline{g}} = |q|_{\overline{g}} \equiv 0. \tag{3.4}$$

In fact

$$\mu = |J|_g \equiv 0,$$

as can be seen from the identity

$$\mu - J(w) = (\mu - |J|_g) + |J|_g(1 - |w|_g) + (|J|_g|w|_g - J(w)),$$

combined with the dominant energy condition (1.1) and the inequality $|w|_g < 1$, which is valid away from ∂M. It then follows from (2.3) and (2.5) that $\overline{R} \geq 2|\overline{E}|^2_{\overline{g}}$. The arguments of [3] may now be used to show that $(\Sigma, \overline{g}, \overline{E})$ coincides with initial data from the $t = 0$ slice of the Reissner-Nordström spacetime, and that ϕ as chosen in (3.3) must be the warping factor of this spacetime. Since $g = \overline{g} - \phi^2 df^2$, the map $x \mapsto (x, f(x))$ yields an isometric embedding of (M, g) into the Reissner-Nordström spacetime. Moreover since $h = k$, a calculation [1, 2] guarantees that the second fundamental form of this embedding agrees with k. Lastly, it is shown in [3] that the electric field E must coincide with the induced electric field of this embedding.

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